

Quantifying the complexity of random Boolean networks

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We study two measures of the complexity of heterogeneous extended systems, taking random Boolean networks as prototypical cases. A measure defined by Shalizi et al. for cellular automata, based on a criterion for optimal statistical prediction [1], does not distinguish between the spatial inhomogeneity of the ordered phase and the dynamical inhomogeneity of the disordered phase. A modification in which complexities of individual nodes are calculated yields vanishing complexity values for networks in the ordered and critical regimes and for highly disordered networks, peaking somewhere in the disordered regime. Individual nodes with high complexity are the ones that pass the most information from the past to the future, a quantity that depends in a nontrivial way on both the Boolean function of a given node and its location within the network.

I. INTRODUCTION

In computational mechanics, the complexity of a process that generates a single time series is defined as the least amount of information required for a maximally accurate statistical description of the series [2–4]. This definition classifies random processes, as well as simple periodic ones, as having low complexity. In a 2004 article, Shalizi et al. extended the definition to spatially-extended dynamical systems and introduced an algorithm for measuring the complexity of a discrete system given time series data for all components [1]. Applying the algorithm to 2D cyclic cellular automata (CCA) confirmed that it classified CCA rules generating fixed states or incoherent local oscillations as having low complexity and cases that produce turbulent spiral waves as having high complexity.

Shalizi’s complexity measure, C_μ , is defined as the amount of information stored in local causal states, where a causal state is an equivalence class of all past configurations that give rise to the same distribution of future outcomes. A set of causal states can be discerned from time series data for all elements in the system. The local complexity (or “complexity density”) is obtained by considering local spacetime regions consisting of truncated past and future light cones for each node in the system.

We study two complexity measures that differ only in the choice of the ensembles of spacetime points used for averaging the local complexity. One approach considers the ensemble of all spatial points at the same time instant, which corresponds to Shalizi’s C_μ . In computing C_μ , all nodes are given equal weight in determining the probabilities of observing different states. Shalizi et al. used C_μ to investigate self-organization of cellular automata, which are logically and topologically uniform networks of discrete, interacting elements.

A second approach is to assign a complexity to each individual element by averaging over time. The system average of these individual complexities is denoted C_ν . For systems that are statistically homogeneous in time and space, C_μ and C_ν are the same. We suggest that C_ν is the one that is most informative for spatially inhomogeneous systems.

In this paper, we analyze the dependence of C_μ and C_ν on parameters specifying ensembles of random Boolean networks (RBNs) with quenched network topology and logic functions. RBNs were first studied by Kauffman as toy models of gene regulatory networks [5, 6]. They have garnered much attention in the last few decades, and features such as steady-state bias, sensitivity to perturbations, attractor length and mutual information between nodes have been extensively investigated [7–10]. We show here that for any given distribution of logic functions, the value of C_μ for a RBN can be analytically calculated as a function of the bias ρ and is not simply related to the sensitivity λ . C_ν on the other hand, is always near zero for sensitivity values $\lambda \leq 1$, where the network is in the ordered or critical regime, and is also zero for the highest possible λ value, where the network dynamics is strongly chaotic. Thus C_ν reflects the intuitive notion that systems with short periodic cycles or apparently random behavior should both have low complexity. The maximum of C_ν for RBNs occurs somewhere in the disordered regime. We find also that the amount of information processed by any given individual depends on global properties of the network dynamics as well as the logic functions of the node in question and others in its neighborhood.

Section II defines the two measures C_μ and C_ν in detail. Section III describes an implementation of these definitions in the context of RBNs and presents theoretical and numerical results on the complexity of a certain class of RBNs. The relation between network complexity and sensitivity, and the relation between individual nodes’ complexity and role in determining the network dynamics is also discussed. We conclude with some general remarks and suggestions for future research.

II. COMPLEXITY MEASURES

The Grassberger-Crutchfield-Young statistical complexity is defined as the least amount of information about the past trajectory required for optimal prediction of future trajectory, given time series data for a single variable [4]. This measure is calculated from time series

data alone, without reference to the physical laws that govern the dynamical processes. Shalizi et al. extended the concept to processes with spatial extent [1]. We summarize the basic ideas here. For details and information-theoretic support of these ideas, see Ref. [11].

Given a field X that varies over space and time in a system where information propagates at a maximum speed of c , the past light cone of a space-time point (\vec{r}, t) consists of all space-time points where events can influence $X(\vec{r}, t)$. $L^-(\vec{r}, t)$ is the configuration of the field X in the past light cone:

$$L^-(\vec{r}, t) = \{(X(\vec{s}, u), t - u, \vec{r} - \vec{s}) \mid \forall u < t \text{ and } |\vec{s} - \vec{r}| \leq c(t - u)\} \quad (1)$$

Note that each element of L^- consists of three quantities: a field value, a time lapse, and a *relative* position. Two space-time points that are influenced by the same past events thus have the same L^- . Similarly, $L^+(\vec{r}, t)$ is the field configuration in the future light cone, the set of points which could be influenced by what happens at (\vec{r}, t) . Each space-time point is associated with one L^- and L^+ . An ensemble of such pairs defines a probability distribution $P(L^+|L^-)$ for future light cone configurations conditioned on past configurations. A causal state $\epsilon(L^-)$ is defined as a set of past light cone configurations that have the same distribution of future configurations. All instances of a given causal state predict the same distribution of future light cone configurations:

$$\epsilon(l^-) = \{\lambda : P(L^+|L^- = \lambda) = P(L^+|L^- = l^-)\}. \quad (2)$$

By definition, $\epsilon(l^-)$ is a sufficient local statistic; knowing the causal state provides the same predictive power as knowing its exact past light cone configuration. $\epsilon(l^-)$ is a *minimal* sufficient statistic [12], meaning that the sufficient statistic $\epsilon(l^-)$ contains the least amount of information among all statistics that have the same predictive power:

$$H[\epsilon(l^-)] \leq H[\eta(l^-)], \quad (3)$$

where $\eta(l^-)$ is a sufficient statistic and $H[X] = -\sum_i P(X = x_i) \log_2 P(X = x_i)$ denotes Shannon entropy. It then follows that $H[\epsilon(l^-)]$ is the least amount of information for optimal prediction of the future dynamics [1, 11], which is taken to be the relevant measure of a system's complexity. We use the shorthand notation

$$C \equiv H[\epsilon(l^-)]. \quad (4)$$

The value of C depends upon the choice of the ensemble of space-time points used to determine the causal states. We consider two choices that are equivalent for spatially homogeneous systems but not for RBNs or other inhomogeneous systems. For C_μ , causal states are determined at any given time by considering the ensemble of spatial locations in the system at that time:

$$\epsilon_\mu(l^-, t) = \{\lambda : P(L^+(\vec{r}, t)|L^-(\vec{r}, t) = \lambda) = P(L^+(\vec{r}, t)|L^-(\vec{r}, t) = l^-)\}. \quad (5)$$

This approach allows one to speak of the complexity of a system as a function of time, which may exhibit transient dynamics. Systems that exhibit an spontaneous increase in $C_\mu(t)$ have been described by Shalizi et al. as going through a self-organization process [1]. For present purposes, we use C_μ to refer to the complexity after transients have decayed.

Alternatively, C_ν is based on causal states defined for each spatially distinct component of the system, using the ensemble of light cone transitions at different times:

$$\epsilon_\nu(l^-, \vec{r}) = \{\lambda : P(L^+(\vec{r}, t')|L^-(\vec{r}, t') = \lambda) = P(L^+(\vec{r}, t)|L^-(\vec{r}, t) = l^-)\}. \quad (6)$$

$C_\nu(\vec{r})$ receives negligible weight from transients, and in practice we compute it by taking data only after transients have relaxed. Because $C_\nu(\vec{r})$ is associated with a particular element of the system, it provides information about the role that element plays in the dynamical evolution of the system. Averaging over all \vec{r} , we obtain a global complexity measure C_ν .

In practice, estimating the complexity requires restricting the depths of the light cone configurations to a manageable size. For the Boolean networks discussed below, we will show that it is also sufficient to consider a single update step in the past and a single step in the future.

III. COMPLEXITY OF RANDOM BOOLEAN NETWORKS

We study $C_\mu(t)$ and $C_\nu(r)$ in synchronously updated RBNs to determine how (or whether) they are related to well-understood measures of the dynamics, such as the sensitivity of the network or the overall bias in the values of the binary variables. In a synchronous RBN, at each time step each node i is updated based on a logic function f_i that is applied to the current values of its input nodes:

$$x_i(t) = f_i(x_{i1}(t-1), x_{i2}(t-1), \dots, x_{in}(t-1)), \quad (7)$$

where t is a positive integer. Here $f_i : \{0,1\}^n \rightarrow \{0,1\}$ is a quenched Boolean logic function for node i , chosen randomly from some fixed distribution over the possible logic functions, and x_{ij} 's are the binary values of the nodes that provide inputs to node i .

The number of inputs and outputs per node, which may or may not be the same for all nodes, and the distribution of logic functions are the parameters that characterize an ensemble of synchronous RBNs. From these parameters, two global measures, the bias and the sensitivity, can be calculated analytically [7, 8]. The bias ρ is the fraction of nodes with value 1 after transients have decayed. Let $\mathbf{x} \in (0,1)^K$ be an input vector of length K , and K be the fixed number of input per node for a network, then the bias map of the network is given by

$$\rho(t+1) = \left\langle \sum_{\mathbf{x}} f(\mathbf{x}) \rho(t)^{|\mathbf{x}|} (1 - \rho(t))^{K-|\mathbf{x}|} \right\rangle, \quad (8)$$

where $\langle \bullet \rangle$ denotes expectation taken over distribution of logic functions and $|\mathbf{x}|$ is the number of 1s in \mathbf{x} . The fixed point or cyclic behavior of the bias is determined by the bias map. The sensitivity λ is the average rate of increase of the Hamming distance between two state space trajectories that initially differ at only a small number of nodes:

$$\lambda = \left\langle \sum_{i=1}^K \sum_{\mathbf{x}} (f(\mathbf{x}^{(i,0)}) \oplus \mathbf{x}^{(i,1)}) \rho^{|\mathbf{x}|} (1-\rho)^{K-|\mathbf{x}|} \right\rangle, \quad (9)$$

where $\mathbf{x}^{(i,j)} = (x_1, \dots, x_{i-1}, j, x_{i+1}, \dots, x_K)$ and \oplus denotes the XOR function [8]. The sensitivity distinguishes qualitatively different network behaviors that have been termed ordered ($\lambda < 1$), disordered ($\lambda > 1$), and critical ($\lambda = 1$) [13].

For the present study, we fix the number of inputs and outputs of the nodes such that the light cones of all nodes have the same shape. We study the simplest nontrivial case, in which all nodes have exactly two inputs and two outputs. For these networks, the greatest possible sensitivity is $\lambda = 2$.

Ref. [1] outlined an algorithm for computationally distinguishing the causal states of cellular automata, and we can implement similar procedures to distinguish the causal states of our RBNs. Because nodes in RBNs are not assigned spatial positions, the definition of L^- and L^+ requires some technical modifications. In the calculation of C_μ , $L_\mu^-(i, t)$ of node i at time step t is defined as

$$L_\mu^-(i, t) = \{(X(j, u), t - u, d(i, j)) \mid \forall u < t \text{ and } d(i, j) \leq t - u\}, \quad (10)$$

where $d(i, j)$ is the shortest distance between nodes i and j . By this definition, different nodes that have the same distance from a given node are deemed indistinguishable for the purpose of identifying a light cone configuration. This is because there is no meaningful way to index the inputs and outputs of nodes for the purpose of comparing light cone configurations of different nodes. Although the update rules (Eq. 7) contain indices for node inputs, the calculation of statistical complexity is done without the knowledge of the underlying physical laws and hence without the knowledge of which input is which at any given node.

For calculating the complexity $C_\nu(i)$ of an individual node, however, different inputs and outputs are distinguishable. When comparing two past or future light cone configurations of the same node i at different time steps, one can keep track of which input is which and observe that when the two inputs differ, the future light cone configuration depends upon which input value is 1. Consequently, $L_\nu^-(i, t)$ is defined as

$$L_\nu^-(i, t) = \{(X(j, u), t - u, j) \mid \forall u < t \text{ and } d(i, j) \leq t - u\}. \quad (11)$$

Following Ref. [1] we limit the depth of light cone to 1. There are two features of a RBN that ensure that light

cones of depth 1 yield the same complexity values as light cones of any other depth. First, there is no memory of previous states in the RBN rules: the configuration at $t = t'$ depends only on what happens at $t = t' - 1$. Secondly, a node's past light cone typically only influences its future light cone through the node itself. Fig. 1 illustrates the crucial difference between a RBN and a regular lattice. In the latter, a node's past light cone can influence its future light cone through multiple paths of the same length as the path passing through the node of interest, whereas in an RBN, the chance of finding such a path is vanishingly small in the limit of large system size. For these reasons, even though the definition of statistical complexity relies on arbitrarily deep light cones, the results for depth 1 become exact for RBNs with system size $N \rightarrow \infty$. This fact enables us to calculate C_μ and, in some cases, C_ν analytically in the large system limit.

C_μ is calculated from depth-1 light cones as follows. For a node with two inputs i and j , there are only three possible past light cone configurations: $\{0, 1\}$, $\{1, 1\}$, and $\{1, 0\}$ or $\{0, 1\}$ (note we have omitted the relative time and distance entries in writing the light cone configurations because they are now trivial after we restrict the light cone depth to 1), occurring with probability $(1-\rho)^2$, ρ^2 , and $2\rho(1-\rho)$, respectively, and yielding probabilities for the reference node being ON of $\langle f(0, 0) \rangle$, $\langle f(1, 1) \rangle$, and $1/2(\langle f(0, 1) \rangle + \langle f(1, 0) \rangle)$ respectively. As discussed above, the probability of observing a future light cone configuration, L^+ , depends only on the state of the reference node. Thus, if each of the three possible L^- 's yields a unique probability for the reference node to be ON or OFF, which in turn yields a unique distribution of L^+ 's, then each L^- is itself a causal state and we have

$$C_\mu = -\rho^2 \log_2(\rho^2) - (1-\rho)^2 \log_2((1-\rho)^2) - 2\rho(1-\rho) \log_2(2\rho(1-\rho)). \quad (12)$$

Modifications to Eq. 12 are required if the above assumptions do not hold. For example, if any two of $\langle f(0, 0) \rangle$, $1/2(\langle f(0, 1) \rangle + \langle f(1, 0) \rangle)$ and $\langle f(1, 1) \rangle$ are equal, then there would be less than three causal states. The following scenario is also possible. If the state of a reference node x is 1, the probability that an output is 1 is

$$\rho_1^+ = \rho \langle f(1, 1) \rangle + 1/2(1-\rho) \langle f(1, 0) \rangle + 1/2(1-\rho) \langle f(0, 1) \rangle; \quad (13)$$

and if $x = 0$, the probability that an output is ON is

$$\rho_0^+ = (1-\rho) \langle f(0, 0) \rangle + 1/2\rho \langle f(1, 0) \rangle + 1/2\rho \langle f(0, 1) \rangle. \quad (14)$$

In the case of an accidental degeneracy $\rho_1^+ = \rho_0^+$, the distribution of L^+ is independent of x and therefore independent of L^- , in which case all three possible L^- collapse to a single causal state, yielding $C_\mu = 0$.

Fig. 2 shows the comparison between the analytical calculation of C_μ and simulation results for ensembles of networks with mixtures of two logic functions. The horizontal axis denotes the fraction of nodes that get

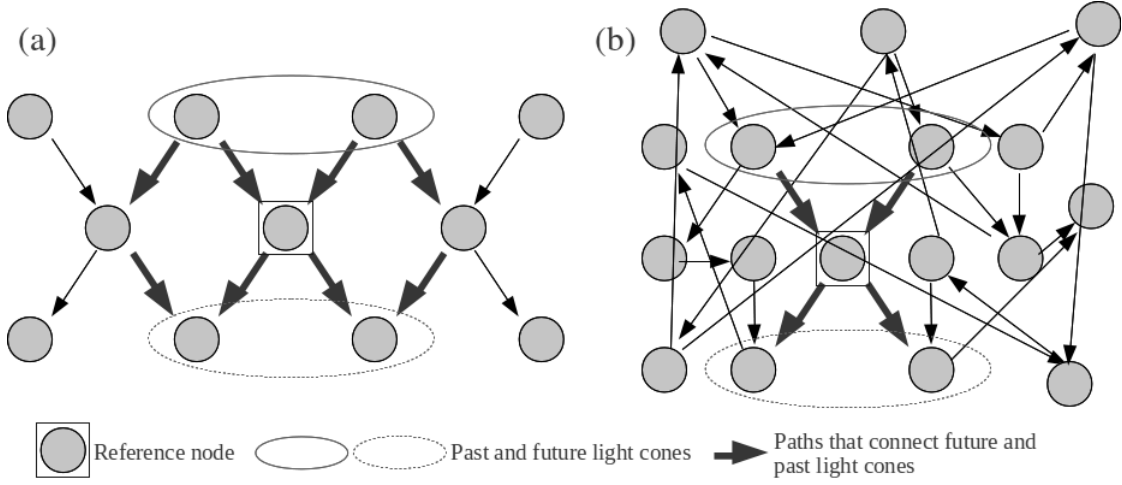


FIG. 1. A regular lattice network and a random network

assigned the indicated Boolean function. The ensemble in Fig. 2(a) satisfies the requirements for Eq. 12. The ensemble in Fig. 2(b) only has two causal states because $\langle f(0,0) \rangle = \langle f(1,1) \rangle = 1 - q$, and $1/2(\langle f(0,1) \rangle + \langle f(1,0) \rangle) = 1$, where q is the fraction of XOR nodes in a XOR-ON network. The first two terms in Eq. 12 collapse into one for such an ensemble, and the corresponding C_μ is given by

$$C_\mu = -(\rho^2 + (1 - \rho)^2) \log_2(\rho^2 + (1 - \rho)^2) - 2\rho(1 - \rho) \log_2(2\rho(1 - \rho)). \quad (15)$$

From Eq. 8 we can obtain the fixed-state bias for a XOR-ON network to be

$$\rho = \frac{-1 + 2q + \sqrt{1 + 4q - 4q^2}}{4q}, \quad (16)$$

which is monotonically decreasing from $\rho = 1$ to $\rho = 0.5$ for $q \in [0, 1]$. We could subsequently obtain a complicated expression of C_μ in terms of q which we would omit here, but we can see that $C_\mu(q)$ is an increasing function in $q \in [0, 1]$ because $C_\mu(\rho)$ monotonically decreases in $\rho \in [0.5, 1]$. Simulation results indicate that C_μ does not change noticeably for a given ensemble for network sizes above $N = 1000$. The simulations for $N = 10^4$ agree well with the large system analytical result. The disagreements at $q = 0.05$ and $q = 1$ in the Fig. 2b are due, respectively, to the fact that differences between causal states are too small for the simulations to resolve ($1 - q$ too close to q) and to a collapse of the type described in the previous paragraph ($\rho_1^+ = \rho_0^+$). For some choices of Boolean functions, the bias ρ can oscillate, which leads to persistent oscillations in C_μ .

Critical networks (with $\lambda = 1$) have been hypothesized to have properties that might be favored by natural selection or other self-organized processes [6]. Our findings show that C_μ is not simply related to sensitivity, so that maximization of C_μ does not correspond to selection of critical networks. Fig. 2 illustrates this point with two

examples. In (a), we see that the network is critical at both $q = 0$ and $q = 0.5$ and that C_μ is a maximum in one case but zero in the other. In (b), we see that C_μ increases monotonically as λ increases from 1 (the critical value) to 2.

The fact that C_μ can be high in ordered systems ($\lambda < 1$), where the attractor dynamics is trivial, highlights the fact that the spatial inhomogeneity of states alone can produce a high complexity. In these networks, almost all nodes are frozen on a fixed value, independent of the initial conditions, but different nodes may be frozen on different values and C_μ becomes a measure of the degree of variation from node to node. The fact that C_μ can be high in strongly disordered systems (λ near the maximum possible value of 2), reflects the tendency of the bias ρ to approach values that maximize C_μ at these points.

The calculation of C_ν treats each component as an agent with its own causal states and complexity, then averages those complexity values. When causal states are determined by considering the past and future light cones of a single node at different times, every frozen node has zero complexity, as does any node for which the past and future light cones are uncorrelated. In ordered or critical networks, where only a vanishingly small number of nodes are not frozen, C_ν is very close to zero. In highly disordered systems, the behavior of most nodes closely approximates a purely stochastic process, so C_ν is again near zero. Thus C_ν is maximized somewhere in the disordered regime.

Fig. 3 shows a typical plot of complexity C_ν as a function of sensitivity λ . Recall that $\lambda = 2$ is the highest sensitivity value possible for a system in which each node has exactly two outputs. All ensembles of systems with a full range of sensitivity values exhibit the same general relation between C_ν and λ , with C_ν maximized at different λ from ensemble to ensemble.

We have not found a way to calculate C_ν analytically for networks with a general combination of logic func-

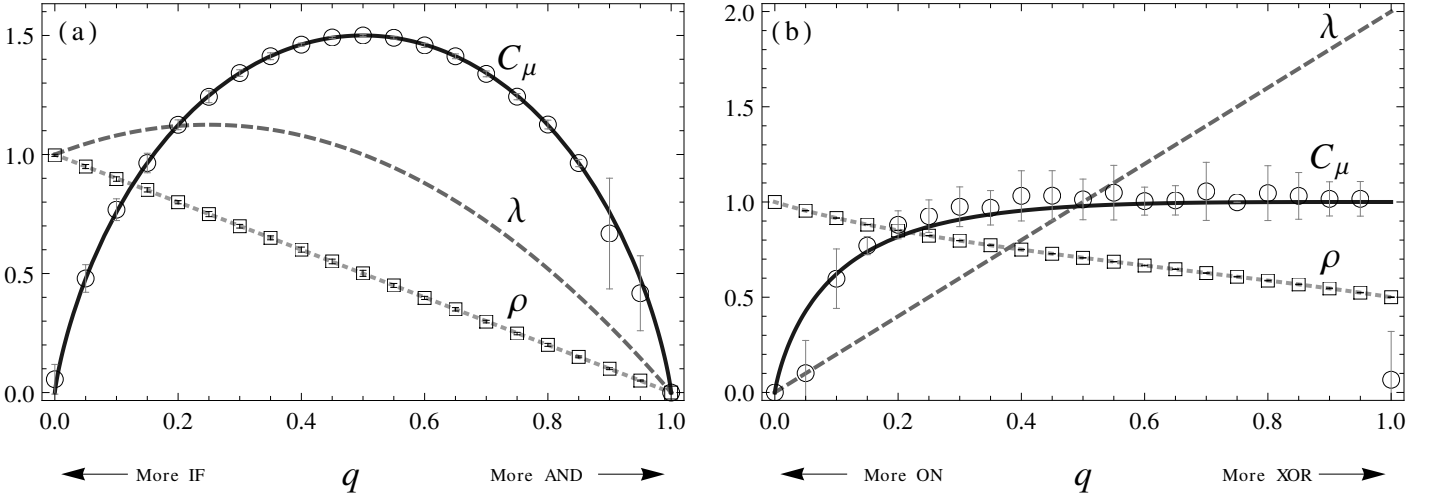


FIG. 2. The steady-state bias ρ , sensitivity λ and complexity C_μ for networks consisting of two types of nodes. (a) A fraction q of the nodes are assigned the AND function ($f(0,0), f(0,1), f(1,0), f(1,1) = (0,0,0,1)$) while the others are assigned $(1,0,1,1)$ (IF). (b) A fraction q are assigned $(0,1,1,0)$ (XOR) and the rest $(1,1,1,1)$ (always ON). The solid curve and circular points are respectively the analytical and simulation results for C_μ . The size of networks is $N = 10^4$, and time of data collection for each network realization is $T = 10^6$. For each of the 21 ratios of the two logic functions (21 dots in the graph), 30 RBNs are constructed. For each RBN, 30 runs are simulated with different initial conditions.

tions, but we can do it (in the large system limit) for the special case of the networks represented in Fig. 2(b), which contain only the logic functions XOR (0110) and ON (1111). In this case, the complexity of each node is either 0 or 1, depending on whether it and/or its neighbors are frozen or not.

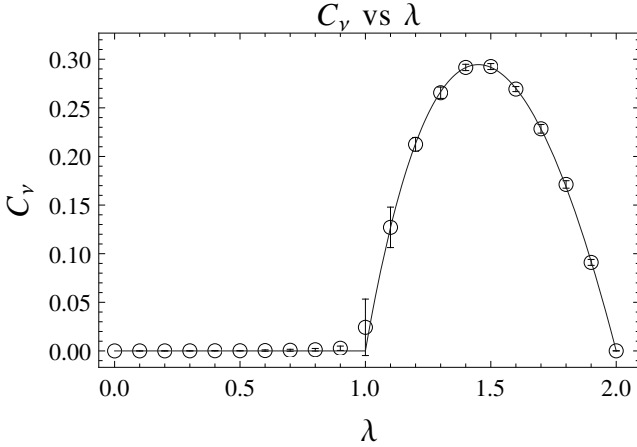


FIG. 3. A typical pattern for complexity C_ν vs sensitivity λ . C_ν is close to zero for $\lambda \leq 1$ and for the maximum value of $\lambda = 2$. C_ν is maximized near $\lambda = 1.5$. The two logic functions are $(0,1,1,0)$ and $(1,1,1,1)$, as in Fig. 2(b). The solid curve shows theoretical results and the circular dots show simulation results. The size of networks is $N = 10^4$ and time of data collection for each network realization is $T = 10^6$. For each of the 21 ratios of the two logic functions (21 dots in the graph), 30 RBNs are constructed. For each RBN, 30 runs are simulated with different initial conditions.

The first step in calculating C_ν is to find the fraction γ of nodes that are frozen. In general, for networks of two-input logic functions in which the probability that a node is frozen when a subset of its inputs are frozen is independent of the value of those frozen inputs, γ is given by a solution of the equation

$$p_0(1-x)^2 + 2p_1x(1-x) + p_2x^2 = x, \quad (17)$$

where p_k is the probability that a node will be frozen if exactly k of its inputs are frozen. (See [14], Eq. (70).) The XOR-ON networks satisfy the requirements and have $p_0 = p_1 = 1 - q$ and $p_2 = 1$, where q is the fraction of XOR nodes, which yields

$$\gamma = (1 - q)/q. \quad (18)$$

Eq. 18 yields values that are greater than 1 for $q < 1/2$. This means the fraction of frozen nodes approaches 1 when network size is large.

The unfrozen nodes are all XORs, some of which have one frozen input and therefore act either as a copier or inverter of the other input. The network of unfrozen nodes thus acts as a network in which every node executes a parity function (or its inversion). Our calculation of C_ν relies on the following conjecture: all nodes in networks consisting of only parity functions (for arbitrary numbers of inputs to each node) will have bias 0.5 when averaged over initial conditions. We further conjecture that the bias is not affected by being embedded in a larger network in which all other nodes freeze after some transient. Our conjectures are supported by numerical simulations of 10,000 randomly generated 16-node networks with only XOR and ON gates. We find that the

bias of 0.5 on each unfrozen node is maintained on each individual time step, not just after averaging over time. In many of these networks, it can be seen that every state of the unfrozen portion has exactly one pre-image, which is enough to guarantee that all states occur with equal frequency when averaged over initial conditions. In cases where some states have two pre-images, the number of recurrent states decreases by a factor of two (and may in some cases be reduced by additional factors of 2). Simulations clearly show that the bias of each unfrozen node averaged over initial conditions is still 0.5 at every time step, but we have not found a rigorous proof that this must be true.

In the large system limit, all frozen nodes have complexity $C_\nu(i) = 0$. To compute C_ν , we must now determine the complexities of the unfrozen nodes. An unfrozen node in the XOR-ON network can have complexity 0 or 1. $C_\nu(i) = 0$ occurs when an unfrozen XOR node i has outputs only to unfrozen nodes that all have additional unfrozen inputs. The value of each output node in this case is equally likely to take either value, no matter what the value of node i . The only way for an unfrozen node i to have nonzero complexity is to have at least one output to a node whose other input is frozen. The future light cone distribution then depends on x_i , and the past light cones that yield the two values occur with equal probability, which gives $C_\nu(i) = 1$.

Calculating C_ν for the XOR-ON ensemble of networks comes down to determining the fraction of nodes that satisfy the condition for having $C_\nu(i) = 1$, which can be obtained from a mean-field calculation as follows. Let X_1 and X_2 be inputs of node X_3 and assume that X_1 has no other output. If we assume that X_1 is an unfrozen node, then the probability that X_2 is frozen and X_3 is unfrozen is $q\gamma$. Because the only way for X_1 to have complexity 1 is for X_2 to be frozen, the probability that X_1 has complexity 1 is $q\gamma$. For an unfrozen node with two outputs, the complexity will be 1 if either of the outputs has a frozen input, which occurs with probability $1 - (1 - q\gamma)^2$. Thus the C_ν is equal to the total fraction of nodes with nonzero complexity:

$$C_\nu = (1 - \gamma)(1 - (1 - q\gamma)^2). \quad (19)$$

Using Eq. 18, we have the system average

$$C_\nu = \frac{1}{q}(1 - q^2)(2q - 1). \quad (20)$$

Eq. 20 agrees well with simulation results (Fig. 3).

The XOR-ON example shows that C_ν can be calculated for specific distributions of logic functions and, more importantly, illustrates that the complexity of an individual component depends on globally determined dynamics, not just on the logical process carried out by the individual node or on the smaller network motifs containing the node. The pattern of frozen nodes is generated by a transient process that may propagate through the entire network [14].

The precise import of the value of $C_\nu(i)$ at a given node is not immediately clear. We have measured the correlation between $C_\nu(i)$ and a new measure $\delta(i)$ that characterizes the effect of replacing node i with a random number generator. $\delta(i)$ is determined as follows. Two realizations of the same network are run in parallel with the same initial condition. After running long enough for transients to decay, the logic function f_i is ignored in one of the copies and node i is replaced by a stochastic agent generating $x_i = 1$ or 0 with probabilities p and $1 - p$, respectively, on each time step. Over the course of M steps, we calculate the fraction of time that each node in the network is ON for each of the realizations, forming two N -dimensional vectors. The Euclidean distance between these two vectors is denoted $\delta(i, p)$, and we define $\delta(i) \equiv \min\{\delta(i, p) : p \in [0, 1]\}$. We obtain an approximate measure of $\delta(i)$ by taking $M = 10N$ and considering $p = 0.1n$ for $n = 0, 1, \dots, 10$. A study of the XOR-AND and IF-AND ensembles shows that the correlation coefficient between $C_\nu(i)$ and $\delta(i)$ in the disordered regime ranges between 0.4 to 0.7 for network size $N=1000$. This suggests an interpretation of $C_\nu(i)$: it is a measure of importance of the logical processing performed at node i for determining the global dynamics. A low value of $C_\nu(i)$ means that the computations done by node i can be effectively simulated by a random number generator.

We note that correlations associated with the existence of two paths in the network from one node to another can cause frozen nodes to have nonzero $C_\nu(i)$. The correlation arises when a path of unfrozen nodes from some node j passes through the frozen node i and then immediately to an unfrozen node k , while another path of unfrozen nodes of the same length goes from j to k without passing through i . Let j_1 be the input to i along the first path. Then x_{j_1} and x_k may be correlated due to the common source at node j , in spite of the fact that node i does not pass on any information. (See Fig. 4.) In the large system limit, the fraction of nodes affected by this type of correlation vanishes.

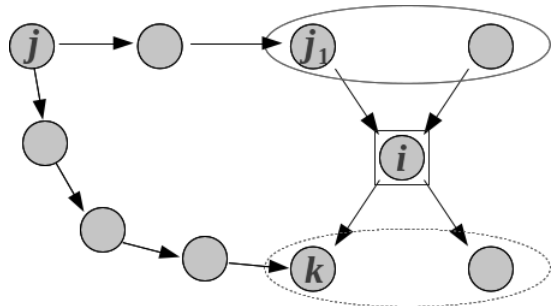


FIG. 4. An example of a network structure that can cause a frozen nodes to have nonzero $C_\nu(i)$: node i is frozen, but its past and future light cones are correlated because nodes j_1 and k receive input from the same node j through chains of unfrozen nodes that differ in length by exactly two links.

Because $\delta(i)$ is necessarily zero for any frozen node, the existence of multiple paths in a finite system causes some nodes with high $C_\nu(i)$ to have low $\delta(i)$. We further note that $\delta(i)$ itself is not a perfect measure of dynamical importance of a given node's activity because the average Euclidean distance between the bias vectors may be small even though the sequence of states is substantially different. In fact, we observe that $\delta(i)$ saturates at low $C_\nu(i)$ values. Comparing $C_\nu(i)$ with more precise measures of dynamical importance would be an interesting topic for future research.

IV. CONCLUSION

In extending the formalism of Shalizi's complexity measure to random Boolean networks, a distinction must be made between determining causal states by averaging over nodes at a given time step versus averaging over time for each node separately. The two methods are equivalent for systems described by the same input-output function at every node of a uniform lattice, but yield different results for systems that are spatially inhomogeneous either because the input-output functions are different for different nodes or because the network topology is not a regular lattice. The networks we have studied have both types of inhomogeneity.

We find that C_μ , obtained from statistics at a single time step, can be calculated analytically for RBN's, and

that it is not simply related to sensitivity to small perturbations. The maximum of C_μ can occur in the ordered, disordered, or critical regime, depending on the details of the probability distribution chosen for the Boolean rules in the network. C_μ is directly related to the steady state (or long-term oscillatory) bias in the node values.

C_ν , obtained from statistics compiled over time for individual nodes, is zero (in the large system limit) everywhere in the ordered and at the limiting value of sensitivity in the disordered regime, so it is maximized somewhere in the disordered regime. The value at an individual node, $C_\nu(i)$, may be interpreted as a measure of the role that node in determining the global dynamics. Nodes that can be replaced by random number generators without substantially altering the global dynamics (and hence perform no essential information processing) tend to have lower $C_\nu(i)$. This last finding may be helpful in characterizing the behavior of social or biological systems and the individual agents or components that comprise them. Interestingly, the $C_\nu(i)$'s depend on the global features of the network that determine its attractors, not just on the local logic functions and topology. In other words, the identification of important players in a network requires global information about the network, not just a characterization of each individual's behavior.

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